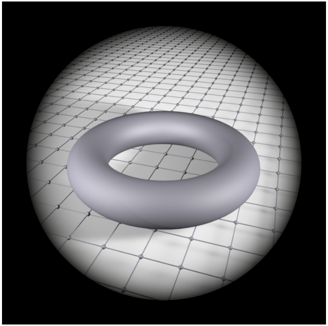
Creating and Probing Topological Matter with Cold Atoms: From Shaken Lattices to Synthetic Dimensions

Nathan Goldman



2015 Arnold Sommerfeld School, August–September 2015

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Outline

Part 1: Shaking atoms!

Generating effective Hamiltonians: “Floquet” engineering

Topological matter by shaking atoms

Some final remarks about energy scales

Part 2: Seeing topology in the lab!

Loading atoms into topological bands

Anomalous velocity and Chern-number measurements

Seeing topological edge states with atoms

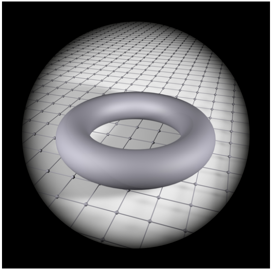
Part 3: Using internal atomic states!

Cold Atoms = moving 2-level systems

Internal states in optical lattices: laser-induced tunneling

Synthetic dimensions: From 2D to 4D quantum Hall effects

Part 3: Using internal atomic states



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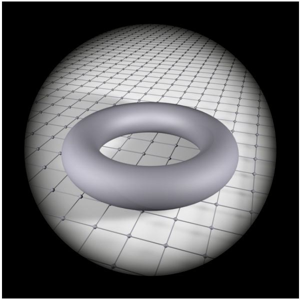
Atoms = moving 2-level systems



- Consider an atom in a laser field (dipole approximation)

$$\hat{H}_{\text{atom}} = \frac{\hat{p}^2}{2M} + \underbrace{\omega_g}_{=0} |g\rangle\langle g| + \sum_j \omega_j |e_j\rangle\langle e_j|,$$

$$\hat{H}_{\text{dip}} = \hat{\boldsymbol{d}} \cdot \boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) + \text{h.c.}, \qquad \hat{\boldsymbol{d}} : \text{dipole operator}, \quad \boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) = E(\boldsymbol{x})\boldsymbol{\varepsilon} \exp(-i\omega_{\text{L}} t)$$

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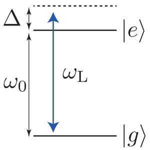
- Simplification : two levels, $|g\rangle$ and $|e\rangle$, entering the problem ($\omega_0 \approx \omega_{\text{L}}$)

$$\hat{H}_{\text{tot}} = \frac{\hat{p}^2}{2M} + \omega_0 |e\rangle\langle e| + \frac{1}{2}\kappa(\boldsymbol{x}) e^{\pm i\omega_{\text{L}} t} |e\rangle\langle g| + \text{h.c.},$$

$$\kappa(\boldsymbol{x}) = 2E(\boldsymbol{x})\, \boldsymbol{\varepsilon} \cdot \langle e|\hat{\boldsymbol{d}}|g\rangle \text{ : Rabi frequency}$$



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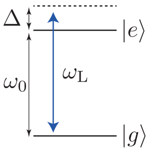

- Effective Hamiltonian (rotating frame at ω_{L} + Rotating Wave Approximation)

$$\hat{H}_{\text{eff}} = \frac{\hat{p}^2}{2M} + \hat{U}_{\text{coupl}}(\boldsymbol{x}), \quad \hat{U}_{\text{coupl}}(\boldsymbol{x}) = \frac{1}{2} \begin{pmatrix} \Delta & \kappa^* \\ \kappa & -\Delta \end{pmatrix}, \quad \Delta(\boldsymbol{x}) = \omega_{\text{L}} - \omega_0(\boldsymbol{x}) : \text{detuning}$$

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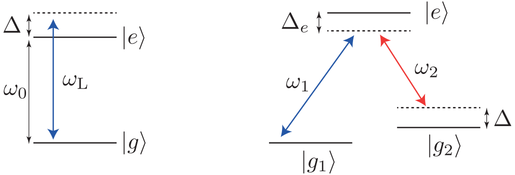

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- Same expression for stimulated Raman transitions between ground-state (Zeeman) sub-levels $|g_1\rangle$ and $|g_2\rangle$: $\kappa = \kappa_1 \kappa_2^* / 2\Delta_e$

- Atom-light coupling Hamiltonian

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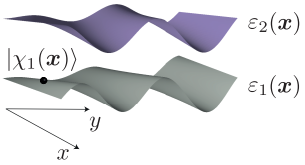
$$\hat{U}_{\text{coupl}}(\boldsymbol{x}) = \frac{\Omega}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix},$$

$$\Omega = \sqrt{\Delta^2 + |\kappa|^2}, \; \cos\theta = \frac{\Delta}{\Omega}, \; \sin\theta = \frac{|\kappa|}{\Omega}, \; \kappa = |\kappa| e^{i\phi}$$

- The eigenvalues are $\varepsilon_{1,2}(\boldsymbol{x}) = \pm\Omega(\boldsymbol{x})/2$ and eigenstates $|\chi_{1,2}(\boldsymbol{x})\rangle$



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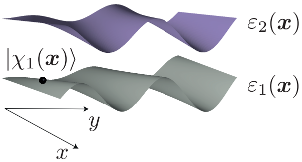

- Born-Oppenheimer approx. ($\Omega \gg$) : we project the dynamics onto a single $|\chi_1(\boldsymbol{x})\rangle$

$$|\Psi(\boldsymbol{x},t)\rangle = \sum_{j=1,2} \psi_j(\boldsymbol{x},t)|\chi_j(\boldsymbol{r})\rangle \approx \psi_1(\boldsymbol{x},t)|\chi_1(\boldsymbol{r})\rangle$$

$$i\partial_t \psi_1(\boldsymbol{x},t) = \left\{ \frac{(\hat{\boldsymbol{p}} - \boldsymbol{A})^2}{2M} + \dots \right\} \psi_1(\boldsymbol{x},t), \qquad \boldsymbol{A} = i\langle \chi_1 | \boldsymbol{\nabla} \chi_1 \rangle = \frac{1}{2}(\cos\theta - 1)\, \boldsymbol{\nabla}\phi$$

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$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{2} \boldsymbol{\nabla}\left(\cos\theta\right) \times \boldsymbol{\nabla}\boldsymbol{\phi}, \qquad \boldsymbol{B} \neq 0 \longrightarrow \boldsymbol{\nabla}\theta \neq 0, \quad \tan\theta = |\kappa|/\Delta$$

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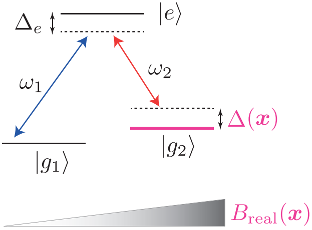

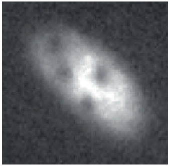

Spielman et al. '09

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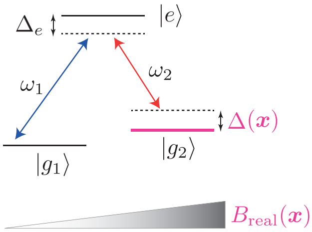
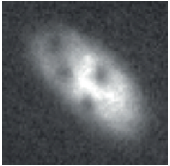
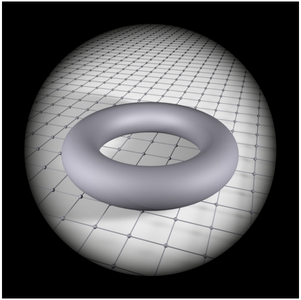




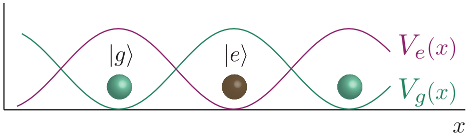
Spielman et al. '09

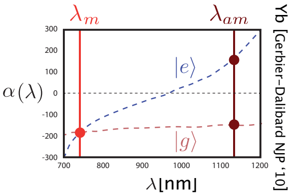
- More dressed states ? One can create spin-orbit coupling $\boldsymbol{A}_{jk} = i\langle\chi_j|\boldsymbol{\nabla}\chi_k\rangle$ ...

Internal states in optical lattices: laser-induced tunneling



- Optical dipole potentials : $V_\sigma(\boldsymbol{x}) = \alpha(\lambda;\sigma)|\boldsymbol{E}(\boldsymbol{x})|^2 \rightarrow$ state-dependent lattices !





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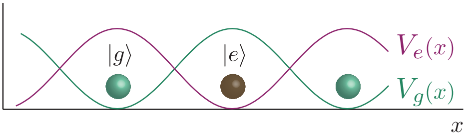

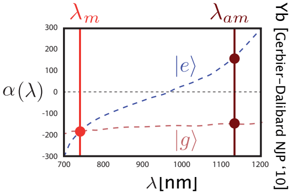

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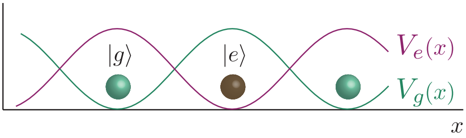

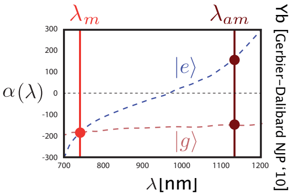

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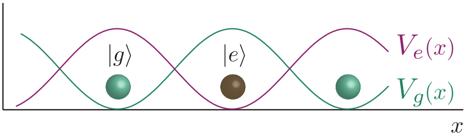

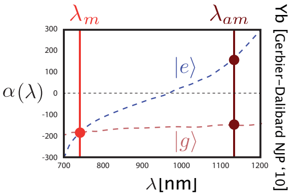

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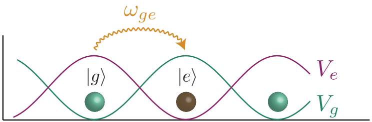

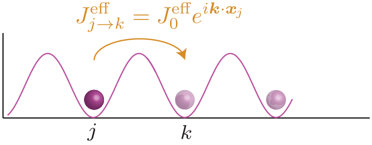

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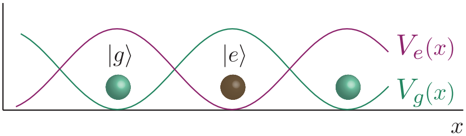

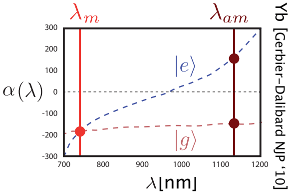

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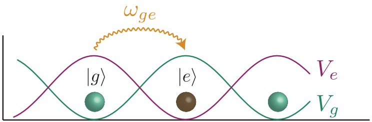
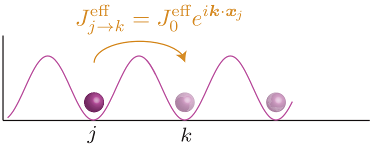


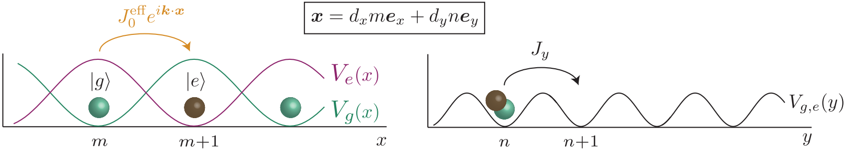


- Jaksch & Zoller [NJP '03] : In the Wannier-states basis $\{|j;g\rangle, |k;e\rangle\}$

$$J_{j\to k}^{\text{eff}} = \langle j;g|\hat{U}_{\text{coupl}}|k;e\rangle = \frac{\Omega}{2}\int w_g(\boldsymbol{x}-\boldsymbol{x}_j)w_e(\boldsymbol{x}-\boldsymbol{x}_k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\text{d}^2x = J_0^{\text{eff}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}_j}$$

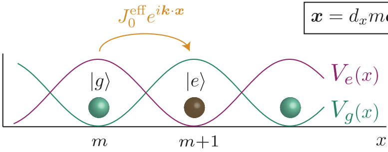
Hopping amplitude : $J_0^{\text{eff}} = \int w_g(\boldsymbol{x}-\boldsymbol{a})w_e(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\text{d}^2x, \quad \boldsymbol{a} = \boldsymbol{x}_k - \boldsymbol{x}_j$

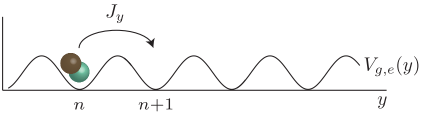
- Let us add a state-**independent** lattice along y



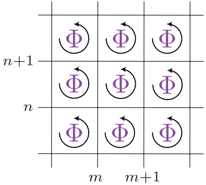
- Let us add a state-**independent** lattice along y

$$\boldsymbol{x} = d_x m \boldsymbol{e}_x + d_y n \boldsymbol{e}_y$$



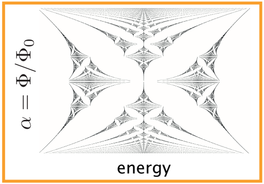


- Setting $\boldsymbol{k} = k_y \boldsymbol{e}_y \rightarrow$ the Harper-Hofstadter model [Jaksch & Zoller, NJP '03]



$$\hat{H} = -J \sum_{m,n} \hat{a}^\dagger_{m,n+1} \hat{a}_{m,n} + e^{i2\pi\alpha n} \hat{a}^\dagger_{m+1,n} \hat{a}_{m,n} + \text{H.c.}$$

$\alpha = \Phi/\Phi_0$: uniform flux per plaquette (in units of flux quantum)



- *Under the carpet here* : the flux had to be rectified [see J-Z, Gerbier-Dalibard '10]
- The synthetic flux is given by $\alpha = k_y d_y / 2\pi \sim 1$

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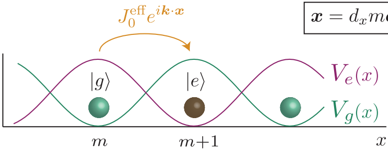

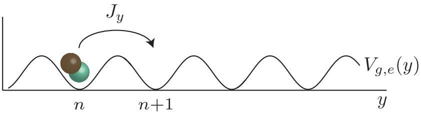

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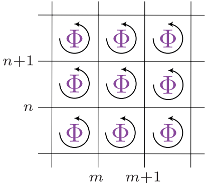

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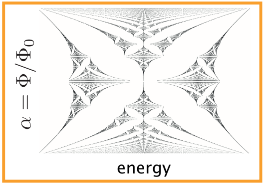

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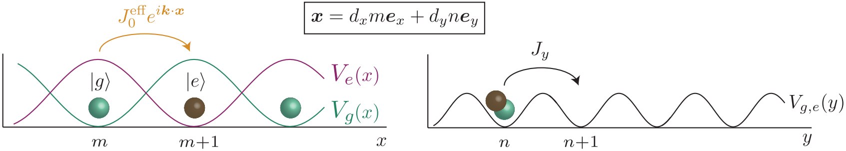
- Reminder : $\alpha \sim 1 \leftrightarrow B \sim 10^4$T in (cond-mat) systems with $d \sim 10^{-10} m$
- The same idea can be used to generate the Haldane model [Anisimovas PRA '14]

- J-Z scheme : state-dependent lattice along x + laser coupling + lattice along y



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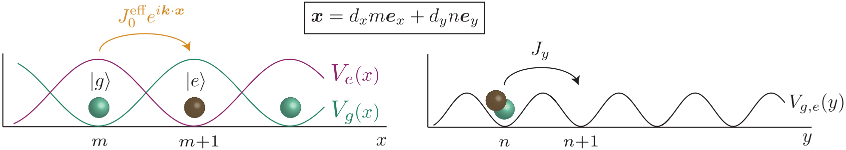

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$$\hat{U}_{\text{coupl}} = \frac{1}{2}\kappa(\boldsymbol{x})|e\rangle\langle g| + \text{h.c.} = \frac{1}{2}\Omega\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}|e\rangle\langle g| + \text{h.c.}, \qquad \text{where we set } E(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

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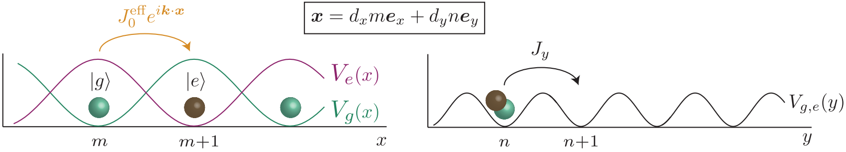

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- We can interpret it as a “hopping” term along the **internal-state dimension**

$$\hat{U}_{\text{coupl}} = J_{\text{synth}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, |1\rangle\langle 2| + \text{h.c.}, \qquad \text{with “hopping” amplitude : } J_{\text{synth}} = \Omega/2$$

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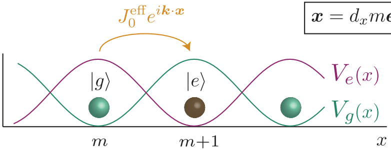

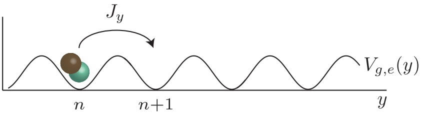

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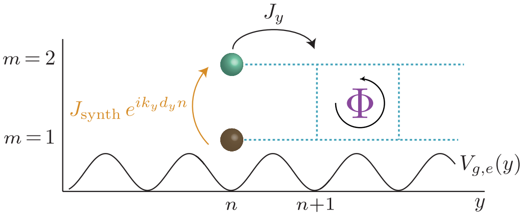
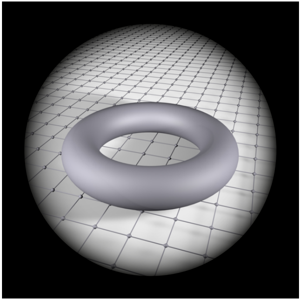
- Let us add a **1D (state-independent) optical lattice** along y and set $\boldsymbol{k} = k_y \boldsymbol{e}_y$:



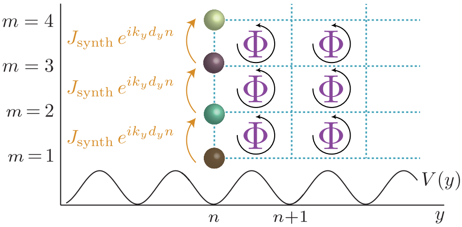
$\longrightarrow$ **2D synthetic ladder** with a synthetic magnetic flux

$$\Phi = 2\pi\alpha = k_y d_y$$

Synthetic dimensions: From 2D to 4D quantum Hall effects



- The **full Harper-Hofstadter lattice** with a synthetic dimension ? More states ?



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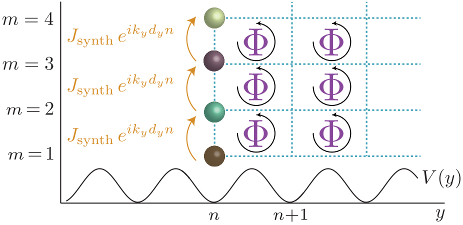

- We have to extend our atom-light problem to $N > 2$ internal states

$$\hat{H}_{\text{atom}} = \frac{\hat{p}^2}{2M} + \underbrace{\omega_g}_{=0} |g\rangle\langle g| + \sum_{j=1}^{N-1} \omega_j |e_j\rangle\langle e_j|,$$

$$\hat{H}_{\text{dip}} = \hat{\boldsymbol{d}} \cdot \boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) + \text{h.c.}, \qquad \hat{\boldsymbol{d}} : \text{dipole operator}, \quad \boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) = E(\boldsymbol{x})\boldsymbol{\varepsilon} \exp(-i\omega_{\text{L}} t)$$

[Goldman, Juzeliunas, Ohberg, Spielman, Rep. Prog. Phys. ’14]

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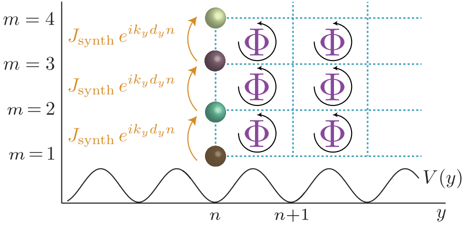

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- Consider Zeeman sublevels $|m_F\rangle$ in the GS manifold (total angular moment. F)
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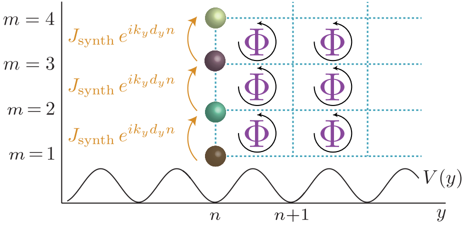

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- Consider Zeeman sublevels $|m_F\rangle$ in the GS manifold (total angular moment. F) Shifted by a **real magnetic field** : $\omega_{j+1} - \omega_j = \delta\omega_0 =$ constant
- A Raman-coupling configuration, with $\omega_1 - \omega_2 = \delta\omega_0$ and $\boldsymbol{k}_1 - \boldsymbol{k}_2 = \boldsymbol{k}_{\text{R}}$, gives

$$\hat{H}_{\text{eff}} = \frac{\Omega_{\text{R}}}{2} \left(\hat{F}_+ e^{i\boldsymbol{k}_{\text{R}} \cdot \boldsymbol{x}} + \hat{F}_- e^{-i\boldsymbol{k}_{\text{R}} \cdot \boldsymbol{x}} \right), \quad \hat{F}_\pm = \hat{F}_x \pm i\hat{F}_y : \text{ ladder operators}$$

$$\hat{F}_+ |m_F\rangle = g_{F,m_F} |m_F + 1\rangle, \quad g_{F,m_F} = \sqrt{F(F+1) - m_F(m_F+1)}$$

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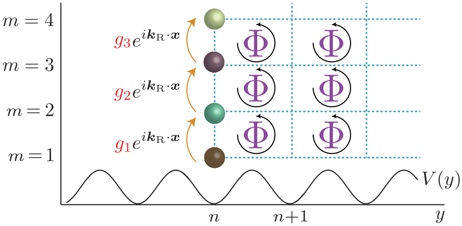

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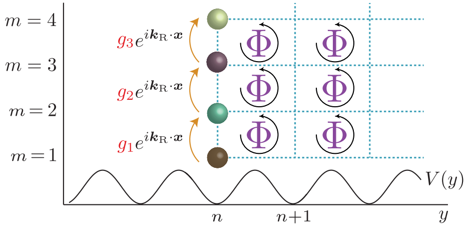

- For $F=1$: $g_{F,m_F} = \sqrt{2} \rightarrow$ isotropic 3-leg ladder with uniform flux !

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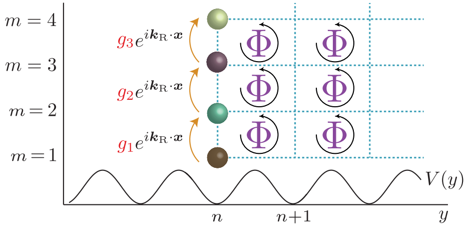
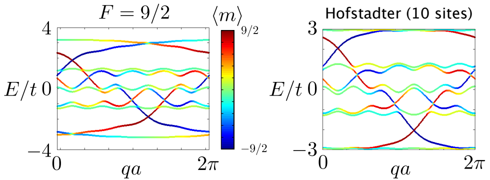
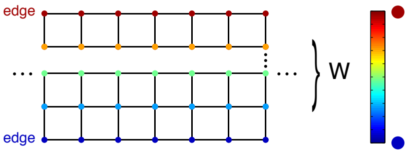


- For $F = 1$: $g_{F,m_F} = \sqrt{2} \to$ isotropic 3-leg ladder with uniform flux !
- For $F = 9/2$ (10-leg ladder) : the anisotropy does not destroy the gaps !

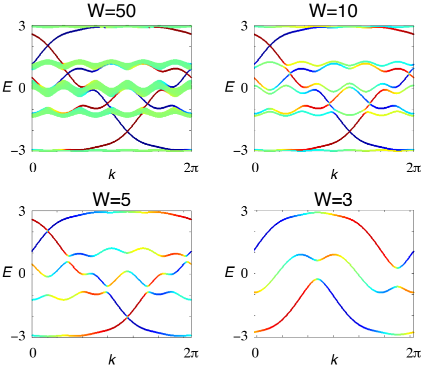


Synthetic lattice and topological edge states

(a) Super-ladder and color code

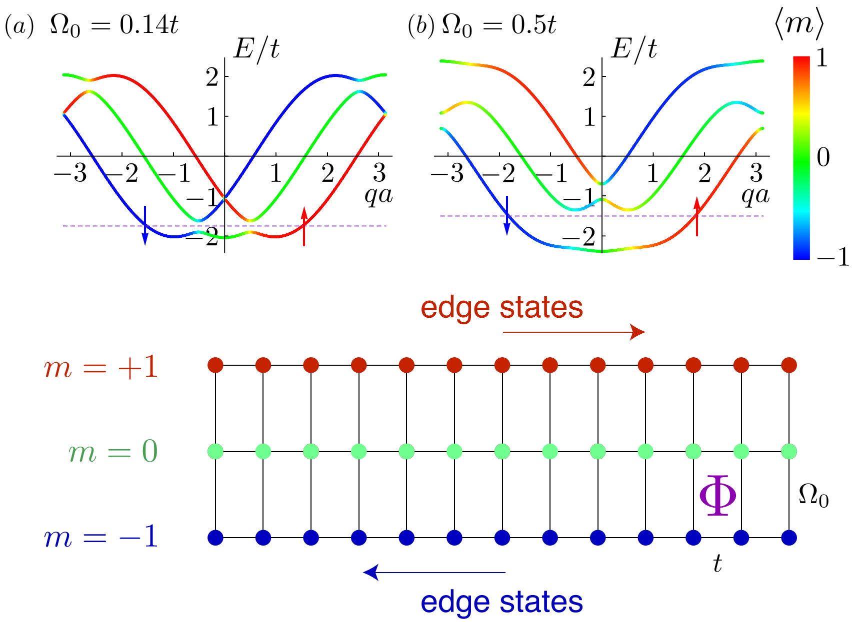


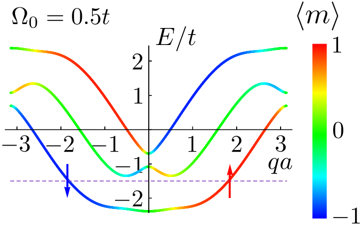
(b) Spectrum



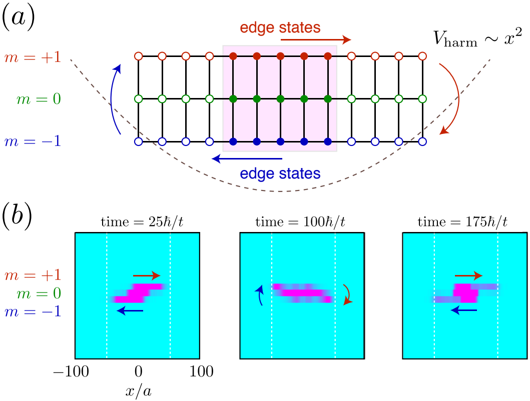
See also D. Hügel and B. Paredes, arXiv :1306.1190 (2013).

Three internal states and the edge states





$\Omega_0 = 0.5t$
E/t
$\langle m \rangle$
qa



(a)
edge states
$V_{\text{harm}} \sim x^2$
$m = +1$
$m = 0$
$m = -1$
edge states
(b)
time = $25\hbar/t$
time = $100\hbar/t$
time = $175\hbar/t$
$m = +1$
$m = 0$
$m = -1$
-100
0
100
x/a

- **Experimental results in 2015 !** arXiv :1502.02495 and arXiv :1502.02496

Observation of chiral edge states with neutral fermions in synthetic Hall ribbons

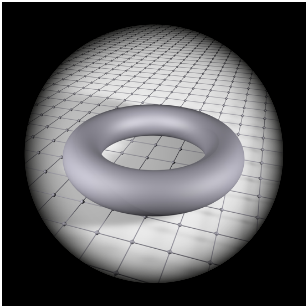
M. Mancini[1], G. Pagano[1], G. Cappellini[2], L. Livi[2], M. Rider[5,6]
J. Catani[3,2], C. Sias[3,2], P. Zoller[5,6], M. Inguscio[4,1,2], M. Dalmonte[5,6], L. Fallani[1,2]
[1] *Department of Physics and Astronomy, University of Florence, 50019 Sesto Fiorentino, Italy*
[2] *LENS European Laboratory for Nonlinear Spectroscopy, 50019 Sesto Fiorentino, Italy*
[3] *INO-CNR Istituto Nazionale di Ottica del CNR, Sezione di Sesto Fiorentino, 50019 Sesto Fiorentino, Italy*
[4] *INRIM Istituto Nazionale di Ricerca Metrologica, 10135 Torino, Italy*
[5] *Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, A-6020 Innsbruck, Austria*
[6] *Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria*

Visualizing edge states with an atomic Bose gas in the quantum Hall regime

B. K. Stuhl[1,*], H.-I Lu[1,*], L. M. Aycock[1,2], D. Genkina[1], and I. B. Spielman[1,†]

[1] Joint Quantum Institute
National Institute of Standards and Technology, and University of Maryland
Gaithersburg, Maryland, 20899, USA
[2] Cornell University
Ithaca, New York, 14850, USA

4D Physics with Cold Atoms



H. M. Price, O. Zilberberg, T. Ozawa, I. Carusotto, N. Goldman, **arXiv:1505.04387**

Beyond the Chern-number measurement...

- What if we combine the electric field E_μ with a perturbing magnetic field $\boldsymbol{B}$?

$$\dot{r}^{\mu}(\boldsymbol{k}) = \frac{\partial \mathcal{E}(\boldsymbol{k})}{\partial k_\mu} - \dot{k}_\nu \Omega^{\mu\nu}(\boldsymbol{k}) \qquad (1)$$

$$\dot{k}_\mu = -E_\mu - \dot{r}^{\nu} B_{\mu\nu}; \quad B_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

see Xiao et al. RMP '10, Gao et al. arXiv :1411.0324

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- Let us insert $\dot{k}_\mu$ into (1) :

$$\begin{aligned}\dot{r}^\mu(\boldsymbol{k}) &= \frac{\partial \mathcal{E}(\boldsymbol{k})}{\partial k_\mu} + E_\nu \Omega^{\mu\nu}(\boldsymbol{k}) + \dot{r}^\gamma B_{\nu\gamma} \Omega^{\mu\nu}(\boldsymbol{k}) \\ &= \frac{\partial \mathcal{E}(\boldsymbol{k})}{\partial k_\mu} + E_\nu \Omega^{\mu\nu}(\boldsymbol{k}) + \left(\frac{\partial \mathcal{E}(\boldsymbol{k})}{\partial k_\gamma} + E_\delta \Omega^{\gamma\delta}(\boldsymbol{k}) + \dot{r}^\alpha B_{\delta\alpha} \Omega^{\gamma\delta}(\boldsymbol{k}) \right) B_{\nu\gamma} \Omega^{\mu\nu}(\boldsymbol{k}) \\ &\approx \frac{\partial \mathcal{E}(\boldsymbol{k})}{\partial k_\mu} + E_\nu \Omega^{\mu\nu}(\boldsymbol{k}) + \frac{\partial \mathcal{E}(\boldsymbol{k})}{\partial k_\gamma} B_{\nu\gamma} \Omega^{\mu\nu}(\boldsymbol{k}) + \Omega^{\gamma\delta}(\boldsymbol{k}) \Omega^{\mu\nu}(\boldsymbol{k}) E_\delta B_{\nu\gamma} + \dots\end{aligned}$$

$\rightarrow$ Combining $\boldsymbol{E}$ and $\boldsymbol{B}$ produces a term $\sim \Omega^2$

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&\approx \frac{\partial \mathcal{E}(\boldsymbol{k})}{\partial k_\mu} + E_\nu \Omega^{\mu\nu}(\boldsymbol{k}) + \frac{\partial \mathcal{E}(\boldsymbol{k})}{\partial k_\gamma} B_{\nu\gamma} \Omega^{\mu\nu}(\boldsymbol{k}) + \Omega^{\gamma\delta}(\boldsymbol{k}) \Omega^{\mu\nu}(\boldsymbol{k}) E_\delta B_{\nu\gamma} + \dots
\end{aligned}$$

$\rightarrow$ Combining $\boldsymbol{E}$ and $\boldsymbol{B}$ produces a term $\sim \Omega^2$

- This raises two questions :
  - What if we fill the band ? Is there (still) a quantized response ?
  - Is there a topological invariant $\int \Omega^2 = \int \Omega \wedge \Omega$?

Some hints from mathematics... see the book by Nakahara

- The curvature is a two-form

$$\Omega = \frac{1}{2}\Omega^{\mu\nu}\mathsf{d}k_\mu \wedge \mathsf{d}k_\nu \qquad \neq 0 \text{ for } \dim(\mathcal{M}) \geq 2$$

- Taking the square produces a four-form

$$\Omega^2 = \Omega \wedge \Omega = \frac{1}{4}\Omega^{\mu\nu}\Omega^{\gamma\delta}\,\mathsf{d}k_\mu \wedge \mathsf{d}k_\nu \wedge \mathsf{d}k_\gamma \wedge \mathsf{d}k_\delta \qquad \neq 0 \text{ for } \dim(\mathcal{M}) \geq 4$$

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$$\Omega = \frac{1}{2}\Omega^{\mu\nu}\mathrm{d}k_\mu \wedge \mathrm{d}k_\nu \qquad \neq 0 \text{ for } \dim(\mathcal{M}) \geq 2$$

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$$\Omega^2 = \Omega \wedge \Omega = \frac{1}{4}\Omega^{\mu\nu}\Omega^{\gamma\delta}\,\mathrm{d}k_\mu \wedge \mathrm{d}k_\nu \wedge \mathrm{d}k_\gamma \wedge \mathrm{d}k_\delta \qquad \neq 0 \text{ for } \dim(\mathcal{M}) \geq 4$$

- Given a curvature Ω, one defines the Chern character

$$\mathrm{ch}(\Omega) = \sum_{j=1} \frac{1}{j!}\mathrm{Tr}\left(\frac{\Omega}{2\pi}\right)^j = \frac{1}{2\pi}\mathrm{Tr}\,\Omega + \frac{1}{8\pi^2}\mathrm{Tr}\,\Omega^2 + \dots$$

  - In 2D : $\mathrm{ch}(\Omega) = \frac{1}{2\pi}\mathrm{Tr}\,\Omega$

  $\longrightarrow$ the first Chern number : $\nu_1 = \frac{1}{2\pi}\int_{\mathcal{M}} \mathrm{Tr}\,\Omega$

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  - In 4D : $\mathrm{ch}(\Omega) = \frac{1}{2\pi}\mathrm{Tr}\,\Omega + \frac{1}{8\pi^2}\mathrm{Tr}\,\Omega^2$

$\longrightarrow$ the second Chern number : $\nu_2 = \frac{1}{8\pi^2}\int_{\mathcal{M}}\mathrm{Tr}\,\Omega^2$

- The second Chern number is associated with the 4D quantum Hall effect

  see Zhang and Hu Science 2001 and Avron et al. PRL 1988 about 4D systems with TRS

Back to the semi-classical equations

- We had the following equations of motion (valid for $d = \text{dim}\mathcal{M} \geq 1$)

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$$\sum_{\boldsymbol{k}} \longrightarrow \frac{V}{(2\pi)^4} \int_{\mathbb{T}^4} \left[1 + \frac{1}{2} B_{\mu\nu} \Omega^{\mu\nu} + \frac{1}{64} \left(\varepsilon^{\alpha\beta\gamma\delta} B_{\alpha\beta} B_{\gamma\delta}\right) \left(\varepsilon_{\mu\nu\lambda\rho} \Omega^{\mu\nu} \Omega^{\lambda\rho}\right)\right] \mathrm{d}^4 k$$

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- The total current density $j^{\mu} = \sum_{\boldsymbol{k}} \dot{r}^{\mu}(\boldsymbol{k})/V$ is given by

$$j^{\mu} = E_{\nu} \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \Omega^{\mu\nu} \mathrm{d}^4 k + \frac{\nu_2}{4\pi^2} \varepsilon^{\mu\alpha\beta\nu} E_{\nu} B_{\alpha\beta} \quad (\mu = x, y, z, w)$$

where $\nu_2 = \frac{1}{8\pi^2} \int_{\mathbb{T}^4} \Omega^2 = \frac{1}{4\pi^2} \int_{\mathbb{T}^4} \Omega^{xy}\Omega^{zw} + \Omega^{wx}\Omega^{yz} + \Omega^{zx}\Omega^{yw} \mathrm{d}^4 k$

In agreement with the topological-field-theory of Qi, Hughes, Zhang PRB '08 for 4D TRS systems

Introducing a 4D framework

- We want to investigate the transport equation

$$j^\mu = E_\nu \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \Omega^{\mu\nu} \mathrm{d}^4 k + \frac{\nu_2}{4\pi^2} \varepsilon^{\mu\alpha\beta\nu} E_\nu B_{\alpha\beta}$$

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- In order to have $\nu_2 \neq 0$, we look for a minimal 4D system with $\Omega^{zx}, \Omega^{yw} \neq 0$

  $\longrightarrow$ fluxes $\Phi_{1,2}$ in the $x-z$ and $y-w$ planes : two Hofstadter models.

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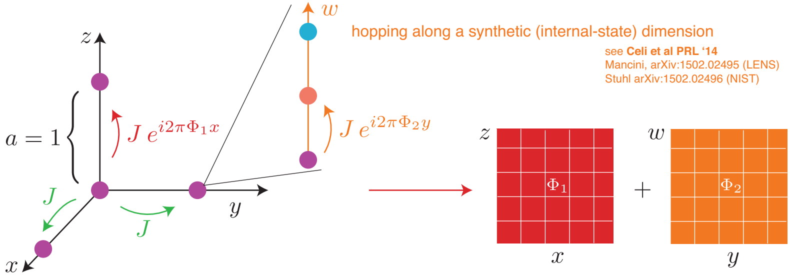

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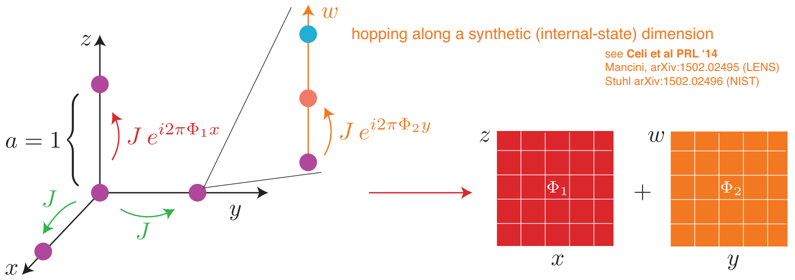
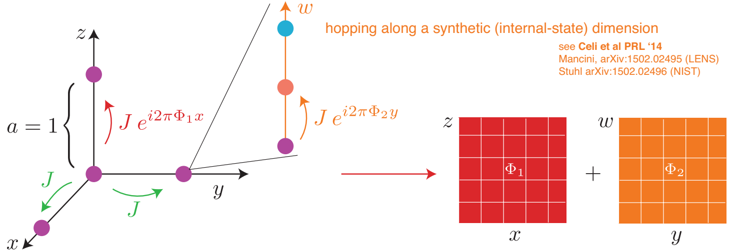
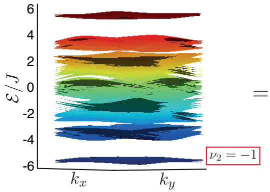


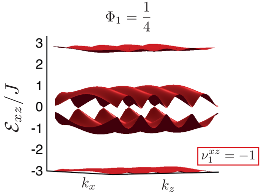
- Physical realization with cold atoms in a 3D optical lattice : Easy !
  - A superlattice along z + resonant $x-z$-dependent time-modulation
  - Raman transitions between internal states with recoil momentum along y

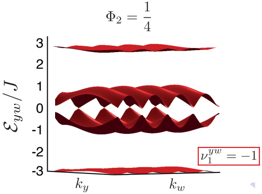


- The energy spectrum displays a low-energy topological band [see Kraus et al. PRL '13]



$$= \mathcal{E}_{xz} + \mathcal{E}_{yw}$$





The transport equations

- Let us come back to our transport equation, with $\Omega^{zx}, \Omega^{yw} \neq 0$

$$j^\mu = E_\nu \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \Omega^{\mu\nu} \mathrm{d}^4 k + \frac{\nu_2}{4\pi^2} \varepsilon^{\mu\alpha\beta\nu} E_\nu B_{\alpha\beta}, \quad \nu_2 = \frac{1}{4\pi^2} \int_{\mathbb{T}^4} \Omega^{zx} \Omega^{yw} \mathrm{d}^4 k$$

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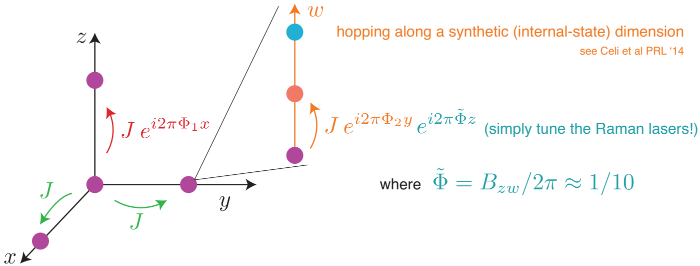
- We now choose an electric field $\boldsymbol{E} = E_y \mathbf{1}_y$ and a magnetic field $B_{\alpha\beta} = B_{zw}$



where $\tilde{\Phi} = B_{zw}/2\pi \approx 1/10$

- The transport equations yield two non-trivial contributions :

$$j^w = E_y \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \Omega^{wy} \mathrm{d}^4 k \text{ : linear response along } w \text{ } (\sim \text{2D QH effect})$$

$$j^x = \frac{\nu_2}{4\pi^2} E_y B_{zw} \text{ : non-linear response along } x \text{ } (\sim \text{4D QH effect})$$

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$$= \frac{E_y}{2\pi} \nu_1^{wy} \times \frac{1}{q} \quad \text{for a flux } \Phi_1 = \Phi_{xz} = p/q.$$

$\longrightarrow \sigma_{\mathrm{H}} = j^w / E_y = \left(\frac{e^2}{h} \right) \frac{\nu_1^{wy}}{q}$: "fractional" Hall conductivity in the $y-w$ plane

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- Could we test all these predictions ?

Numerical simulations : the current density

- The transport equations yield two non-trivial contributions for E_y and B_{zw} :

$$j^w = \frac{E_y}{2\pi} \nu_1^{wy} \times \frac{1}{q} \qquad \text{for a flux } \Phi_1 = \Phi_{xz} = p/q$$

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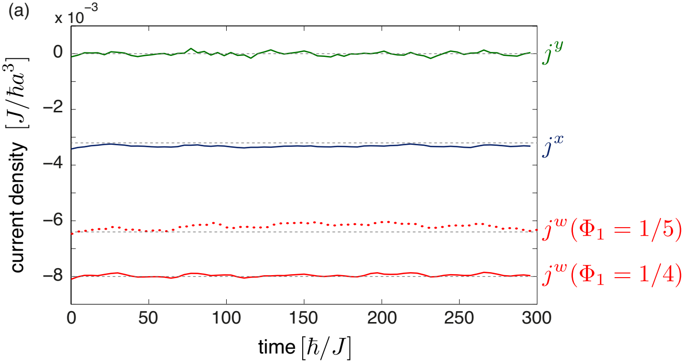
- We have calculated the current densities for $E_y = -0.2J/a$ and $B_{zw}/2\pi = -1/10$



- From these simulations : $\nu_2 \approx -1.07$ and $\nu_1^{wy} \approx -1.03$

The center-of-mass drift : Numerical simulations

- The predicted center-of-mass drift along x (2nd-Chern-number response) :

$$v^x_{\text{c.m.}} = j^x A_{\text{cell}} = j^x \left(4a \times 4a \times a \times a\right), \quad \text{for } \Phi_1 = \Phi_2 = 1/4$$

$$= \left(\frac{\nu_2}{4\pi^2} E_y \times B_{zw}\right) \times 16a^4 \approx 2a/T_B, \qquad T_B = 2\pi/aE_y \approx 50\text{ms}$$

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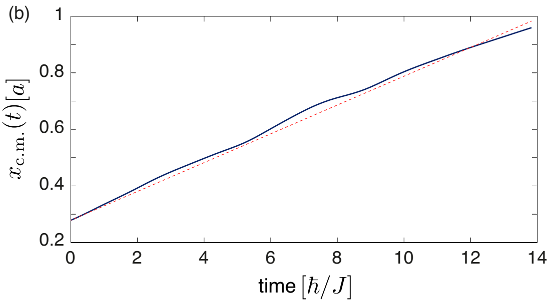
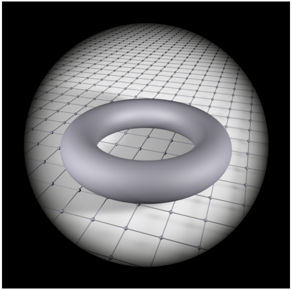
- We have calculated the COM trajectory for $E_y = 0.2J/a$ and $B_{zw}/2\pi = -1/10$



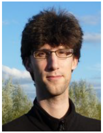
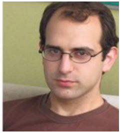
- From these simulations : $\nu_2 \approx -0.98$

The 4D responses are **of the same order**
as the effects reported in *Aidelsburger et al '15*!

Some brief acknowledgements



- **Munich team**: M. Aidelsburger M. Lohse, C. Schweizer, M. Atala, J. T. Barreiro, and I. Bloch

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- **Cambridge**: N. R. Cooper
- **NIST**: I.B. Spielman

    

- M. Aidelsburger M. Lohse, C. Schweizer, M. Atala, J. T. Barreiro, S. Nascimbene, N. R. Cooper, I. Bloch and N. Goldman, **Nature Physics 11, 162–166 (2015)**
- N. Goldman and J. Dalibard, **PRX 4, 031027 (2014)**
- N. Goldman, J. Dalibard, M. Aidelsburger, N. R. Cooper, **PRA 91, 033632 (2015)**
- N. Goldman, J. Dalibard, A. Dauphin, F. Gerbier, M. Lewenstein, P. Zoller, and I. B. Spielman, **PNAS 101, 1 (2013)**
- N. Goldman, G. Juzeliunas, P. Ohberg, I. B. Spielman **Rep. Prog. Phys. 77 126401 (2014)**
- S. Nascimbene, N. Goldman, N. R. Cooper, J. Dalibard **to appear in PRL (2015)**

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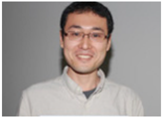
- Oded Zilberberg, **ETH (Zurich)**

Hannah



Tomoki

Iacopo

Oded

- H. M. Price, O. Zilberberg, T. Ozawa, I. Carusotto, N. Goldman, **arXiv:1505.04387**